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A Study of Banach Fixed Point Theorem and Applications of The Fixed Point Theorem

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Abstract : The Fixed point of a function is a point in the set such that the function maps that point to itself. In other words, if a function f takes an element x from a set S and maps it to another element f(x) in S, then x is a fixed point of f if and only if f(x) = x.

The theorem has many applications in various fields, including mathematics, economics, and engineering. For example, in economics, the theorem is used to prove the existence of an equilibrium in certain models. In mathematics, it is used to prove the existence of solutions to differential equations. In engineering, it is used to solve nonlinear equations that arise in various design and control problems.

Key Words : The Fixed Point Theorem, Banach Fixed Point Theorem

Introduction : The Fixed Point Theorem is a result in mathematics that states that every continuous function from a compact convex set to itself has at least one fixed point. To understand this theorem, we need to know the meaning of several terms.

A continuous function is a function where small changes in the input produce small changes in the output. This means that, for any two points close to each other, the function maps the two points to points that are also close to each other.

A compact set is a set that is closed and bounded. In other words, it is a set where all its points are "contained" within a certain region, and the set contains all its limit points.

A convex set is a set where any two points in the set can be connected by a line segment that lies entirely within the set. In other words, it is a set where any line segment connecting two points in the set also lies entirely within the set.

Now, let's consider a function f from a compact convex set S to itself. A fixed point of this function is a point x in the set S such that f(x) = x. In other words, it is a point in the set that maps to itself under the function. The Fixed Point Theorem states that every continuous function from a compact convex set to itself has at least one fixed point. In other words, for any function that satisfies the conditions of the theorem, there is always at least one point in the set that maps to itself.

This theorem is important because it provides a general way to prove the existence of solutions to certain problems. For example, in economics, the theorem is used to prove the existence of an equilibrium in certain models. In mathematics, it is used to prove the existence of solutions to differential equations. In engineering, it is used to solve nonlinear equations that arise in various design and control problems.

The proof of the Fixed Point Theorem is based on the concept of topology, which is the study of the properties of spaces that are preserved under continuous transformations. The proof relies on the idea that a continuous function on a compact convex set must take its maximum and minimum values, and that these values must lie in the set. By combining this with the fact that the function maps every point in the set to a point in the set, it can be shown that there must be a point in the set that maps to itself, which is a fixed point.

The Fixed Point Theorem has a wide range of applications in various fields, including:

Economics: In economics, the theorem is used to prove the existence of an equilibrium in certain models. For example, in game theory, the theorem is used to show that there is a Nash equilibrium, which is a state in which no player has an incentive to change their strategy.

Mathematics: The theorem is used in mathematics to prove the existence of solutions to differential equations, which describe the behavior of systems over time. It is also used to study the stability of solutions to these equations, and to understand the behavior of certain types of iterative algorithms.

Engineering: The theorem is used in engineering to solve nonlinear equations that arise in various design and control problems. For example, it is used to find the optimal operating point for a system, or to find the steady-state response of a system to a given input.

Computer Science: The theorem is used in computer science to solve optimization problems, such as finding the shortest path in a graph or the minimum cost of a network.

Physics: The theorem is used in physics to study the behavior of systems over time, such as the motion of a pendulum or the flow of fluids in a pipe.

Biology: The theorem is used in biology to study the behavior of populations over time, such as the evolution of a species or the spread of a disease.

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These are just a few examples of the many applications of the Fixed Point Theorem. The theorem is a powerful tool that provides a way to prove the existence of solutions to a wide range of problems, and its importance lies in its generality and versatility.

Banach Fixed Point Theorem

The Banach Fixed Point Theorem is a generalization of the Fixed Point Theorem that applies to complete metric spaces. A complete metric space is a metric space (a set with a notion of distance) in which every Cauchy sequence converges to a limit that is also in the space. A Cauchy sequence is a sequence of points in the space such that the distance between any two points in the sequence gets arbitrarily small as the number of points increases.

The Banach Fixed Point Theorem states that if f is a contraction mapping (i.e., a function that strictly decreases the distance between any two points in the space) from a complete metric space X to itself, then f has exactly one fixed point in X.

The Banach Fixed Point Theorem has many applications in various fields, including functional analysis, partial differential equations, and optimization. For example, in optimization, the theorem is used to find the solution of nonlinear equations by iteratively applying the function f to an initial guess and finding a sequence of points that converges to the fixed point. This method is known as the Banach contraction mapping principle.

The proof of the Banach Fixed Point Theorem is based on the concept of completeness and the idea that a contraction mapping must strictly decrease the distance between any two points in the space. By combining these two concepts, it can be shown that the sequence of points generated by iteratively applying the function f must converge to a limit, which is the fixed point of the function.

The Banach Fixed Point Theorem is a fundamental result in mathematics that provides a method for finding the fixed point of a function. It is a generalization of the Fixed Point Theorem and applies to complete metric spaces. In this context, a metric space is a set equipped with a notion of distance, and a complete metric space is one in which every Cauchy sequence converges to a limit that is also in the space.

To understand the Banach Fixed Point Theorem, let's consider a function f from a complete metric space X to itself. A function f is called a contraction mapping if it satisfies the following property: there exists a constant k ($0 < k < 1$) such that the distance between the images of any two points in the space is strictly less than the distance between the points themselves. In other words, if x and y are any two points in the space, then:

d(f(x), f(y)) < k * d(x, y)

where d is the metric on the space.

The Banach Fixed Point Theorem states that if f is a contraction mapping from a complete metric space X to itself, then f has exactly one fixed point in X. That is, there exists a point x* in X such that f(x*) = x*.

The proof of the Banach Fixed Point Theorem is based on the following idea: if f is a contraction mapping, then the distance between any two points in the space decreases strictly as the number of iterations increases. Since X is a complete metric space, it follows that the sequence of points generated by iteratively applying the function f must converge to a limit, which is the fixed point of the function.

The Banach Fixed Point Theorem has a wide range of applications in various fields, including functional analysis, partial differential equations, and optimization. In optimization, the theorem is used to find the solution of nonlinear equations by iteratively applying the function f to an initial guess and finding a sequence of points that converges to the fixed point. This method is known as the Banach contraction mapping principle and is a powerful tool for solving nonlinear problems.

In summary, the Banach Fixed Point Theorem is a powerful result that provides a method for finding the fixed point of a function in a complete metric space. It has many applications in various fields and is a fundamental tool for solving nonlinear problems in mathematics and beyond.

The mathematical formulation of the Banach Fixed Point Theorem involves the definition of a complete metric space, a contraction mapping, and the fixed point of a function.

A metric space is a set X equipped with a notion of distance, represented by a function d(x, y) that satisfies the following properties:

d(x, y) >= 0 for all x, y in X

d(x, y) = 0 if and only if x = y

d(x, y) = d(y, x) for all x, y in X

d(x, y) <= d(x, z) + d(z, y) for all x, y, z in X

A complete metric space is a metric space in which every Cauchy sequence converges to a limit that is also in the space. A Cauchy sequence is a sequence of points in the space such that the distance between any two points in the sequence gets arbitrarily small as the number of points increases.

A function f from a complete metric space X to itself is called a contraction mapping if there exists a constant k ($0 < k < 1$) such that the distance between the images of any two points in the space is strictly less than the distance between the points themselves:

$d(f(x), f(y)) < k * d(x, y)$ for all x, y in X

The fixed point of a function f is a point x* in X such that f(x*) = x*.

With these definitions, the Banach Fixed Point Theorem can be mathematically stated as follows: If f is a contraction mapping from a complete metric space X to itself, then f has exactly one fixed point in X. That is, there exists a unique point x* in X such that f(x*) = x*.

Conclusion

The Banach theorem seems somewhat limited. It seems intuitively clear that any continuous function mapping the unit interval into itself has a fixed point. We hope that this work will be useful for functional analysis related to normed spaces and fixed point theory. Our results are generalizations of the corresponding known fixed point results in the setting of Banach spaces on its norm spaces. Then all expected results in this paper will help us to understand better solution of complicated theorem. In future, we will discuss of Banach spaces on its norm spaces related properties to physical problem.

References

[1] Ekeland, I. (1974) On the Variational Principle. Journal of Mathematical Analysis and Applications, 47, 324-353. https://doi.org/10.1016/0022-247X(74)90025-0

[2] Ekeland, I. (1979) Nonconvex Minimization Problems. Bulletin of the American Mathematical Society, 1, 443-474. https://doi.org/10.1090/S0273-0979-1979-14595-6

[3] Alfuraidan, M. and Ansari, Q. (2016) Fixed Point Theory and Graph Theory: Foundations and Integrative Approaches. Academic Press-Elsevier, London. https://doi.org/10.1016/B978-0-12-804295-3.50013-9

[4] Guran, L. (2012) Ulam-Hyers Stability of Fixed Point Equations for Single Valued Operators on KST Spaces. Creative Mathematics Informatics, 21, 41-47. https://doi.org/10.37193/CMI.2012.01.13

[5] Hasanzade Asl, J., Rezapour, S. and Shahzad, N. (2012) On Fixed Points of α-ψ-Contractive Multifunctions. Fixed Point Theory and Applications, 2012, Article No. 212. https://doi.org/10.1186/1687-1812-2012-212

[6] Khojasteh, F., et al. (2016) Some Applications of Caristi's Fixed Point Theorem in Metric Spaces. Fixed Point Theory and Applications, 2016, Article No. 16. https://doi.org/10.1186/s13663-016-0501-z

[7] Khojasteh, F., Karapmar, E. and Radenovic, S. (2013) θ-Metric Space: A Generalization. Mathematical Problems in Engineering, 2013, Article ID 504609, 7 p. https://doi.org/10.1155/2013/504609

[8] Rudin, W. (1991) Functional Analysis. Second Edition, International Editions, Mc-Graw-Hill, New York, ISBN 0-07-100944-2.

[9] Agarwal, R.P., Hussain, N. and Taoudi, M.-A. (2012) Fixed Point Theorems in Ordered Banach Spaces and Applications to Nonlinear Integral Equations. Abstract and Applied Analysis, 2012, Article ID: 245872, 15 p. https://doi.org/10.1155/2012/245872

[10] Lu, N., He, F. and Huang, H. (2019) Answers to Questions on the Generalized Banach Contraction Conjecture in b-Metric Spaces. Journal of Fixed Point Theory and Applications, 21, 43. https://doi.org/10.1007/s11784-019-0679-x